Chern–Simons theory of multi-component quantum Hall systems

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The Chern–Simons approach has been widely used to explain fractional quantum Hall states in the framework of trial wave functions. In the present paper, we generalise the concept of Chern–Simons transformations to systems with any number of components (spin or pseudospin degrees of freedom), extending earlier results for systems with one or two components. We treat the density fluctuations by adding auxiliary gauge fields and appropriate constraints. The Hamiltonian is quadratic in these fields and hence can be treated as a harmonic oscillator Hamiltonian, with a ground state that is connected to the Halperin wave functions through the plasma analogy. We investigate conditions on the coefficients of the Chern–Simons transformation and on the filling factors under which our model is valid. Furthermore, we discuss several singular cases, associated with states with ferromagnetic properties.

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I. INTRODUCTION

In understanding the fractional quantum Hall effect (FQHE), the now famous trial wave function proposed by Laughlin[1] proved to be a successful approach to describe the physics of incompressible quantum liquids at certain fractional filling factors. Laughlin's wave function is furthermore the inevitable starting point for several generalisations, such as Jain's composite-fermion proposal,[2,3] Halperin's two-component wave function,[4] or more complicated wave functions describing states possessing quasi-particle excitations with nonabelian statistics.[5]

A field-theoretical approach, complementary to the above-mentioned one, consists of so-called Chern–Simons theories, which formalise the idea of flux attachment that is also implicit in the trial wave functions. Chern–Simons theories have been successfully elaborated to study incompressible[6] and compressible[7] quantum liquids in one-component systems as well as two-component systems,[8–12] which comprise e.g. bilayer quantum Hall systems or single layer systems in situations where the spins are not completely polarised. Multi-component Chern–Simons approaches have also been proposed in the study of edge excitations of the incompressible quantum Hall liquids.[13] An undeniable advantage of these Chern–Simons theories consists of their transparent insight into the exotic properties of these quantum liquids, such as their topological degeneracy, the fractional charges of their quasi-particle excitations or the statistical properties of the latter.[8,9] However, the Chern–Simons theories are usually less adapted when it comes to calculating quantities involving energy scales. Indeed, Chern–Simons transformations act on the kinetic part of the electronic Hamiltonian, whereas they leave the interaction part invariant. The kinetic part gets therefore renormalised but continues to determine the overall energy scale whereas the physical energy scale in the FQHE must be set by the electron-electron interactions.

A successful generalisation of Chern–Simons theories, that does not suffer from the problem of the correct energy scale, is the Hamiltonian theory proposed by Shankar and Murthy.[14–18] This theory is a very powerful tool for the computation of physical quantities,[17,18] and even for the description of higher-generation composite fermion states.[19] However, it is limited by the fact that it does not incorporate internal degrees of freedom. The success of the single-component Hamiltonian theory justifies a generalisation that can be applied to describe systems for which internal degrees of freedom (spin and/or pseudospin) are relevant. The main interest in such a generalisation stems from realistic systems with more than two internal degrees of freedom, such as graphene with its four-fold spin-valley degeneracy[20] or bilayer quantum Hall systems with non-polarised electron spins.

In this paper, we analyse a multi-component Chern–Simons theory within the framework of the microscopic theory by Shankar and Murthy.[14–18] This approach has two main advantages over the previously proposed ones. First, it allows one to distinguish between physically relevant Chern–Simons theories from those which are ill-defined. The basic ingredient for this distinction is the $\kappa \times \kappa$ charge matrix K, which was first introduced by Wen and Zee.[8,9] We find that matrices with negative eigenvalues need to be discarded in the study of physically relevant Chern–Simons theories because they would lead to ground-state wave functions that cannot be normalised. This structural feature of Chern–Simons theories finds its physical interpretation within Laughlin's plasma analogy[1] that indicates a tendency of the different components to undergo a phase separation and thus to form spatially inhomogeneous states. We show that zero eigenvalues of the charge matrix K, in contrast to the unphysical negative eigenvalues, find a compelling interpretation in terms of ferromagnetic quantum Hall states. Our results thus generalise previous work on two-component systems by Lopez and Fradkin[10] to an arbitrary number of components κ.

A second advantage of the present approach consists of a transparent connection between multi-component Chern–Simons theories with trial wave functions. It has been shown, in the simpler one-component case, that treating the fluctuations of the Chern–Simons vector potential within the harmonic approximation (Gaussian model) yields Laughlin's and Jain's (unprojected) composite-fermion wave functions.[18,21] Similarly, we obtain here, within the Gaussian model of

κ-component fluctuating Chern–Simons vector potentials, multi-component trial wave functions[22] that are generalisations of Halperin's two-component wave functions.[4,10] Furthermore, we obtain in the same manner composite-fermion-type wave functions that may be viewed as particular multi-component generalisations of Jain's original proposal.[2,3]

The paper is organised as follows. In Sec. II, we define the Chern–Simons transformations for systems with κ components and introduce, in Sec. III, extra degrees of freedom, in the form of the auxiliary gauge fields, as described by Shankar and Murthy. We subsequently diagonalise the harmonic oscillator Hamiltonian and investigate the connection of the resulting wave function with trial wave functions through the plasma analogy. In Sec. IV, we extend our results to the situation of singular K matrices and discuss the relation between residual symmetries and underlying ferromagnetic properties of the quantum Hall states. Our conclusions are presented in Sec. V.

II. CHERN–SIMONS TRANSFORMATIONS

We consider a quantum Hall system with κ internal states, hereafter referred to as "components". In the simplest case of a two-dimensional electron gas at a GaAs/AlGaAs interface, one has $\kappa = 2$ for the two possible orientations of the electron spin. The case $\kappa = 4$ is relevant for bilayer quantum Hall systems, where a second pseudospin mimics the layer index, or in graphene due to its two-fold valley degeneracy, in addition to the physical spin of the electrons. Higher values of κ are rarely discussed in the literature, but may play a role in the context of multilayer systems or of bilayer graphene, where the zero-energy level consists of the $n = 0$ and $n = 1$ Landau levels.[20] The Chern–Simons transformation[2,23] is defined by the relation between the κ original electronic fields $\psi_\alpha(\mathbf{r})$ and the κ transformed fields $\psi_\alpha^{\mathrm{CS}}(\mathbf{r})$ as

$$\psi_\alpha(\mathbf{r}) = \exp\biggl(-i \int \mathrm{d}^2\mathbf{r}'\theta(\mathbf{r}-\mathbf{r}') \sum_{\beta=1}^{\kappa} K_{\alpha\beta}\rho_\beta(\mathbf{r}')\biggr)\psi_\alpha^{\mathrm{CS}}(\mathbf{r}), \tag{2.1}$$

where $\theta(\mathbf{r}) = \arg(x+iy)$ indicates the angle between the vector $\mathbf{r} = (x,y)$ and the $\mathbf{e}_x$ direction, and $\rho_\beta(\mathbf{r}) = \psi_\beta^\dagger(\mathbf{r})\psi_\beta(\mathbf{r}) = \psi_\beta^{\mathrm{CS}\,\dagger}(\mathbf{r})\psi_\beta^{\mathrm{CS}}(\mathbf{r})$ is the density operator of the particles of component β. The $\kappa \times \kappa$ matrix $K_{\alpha\beta}$ encodes the topological properties of the underlying quantum liquids, such as its degeneracy, the charges of its quasi-particle excitations and the statistics of the latter.[8,9] Physically, it indicates the number of flux quanta attached to particles of component α due to the density of particles of component β. This transformation is a *singular* transformation for the reason that $\theta(\mathbf{r}-\mathbf{r}')$ has a singularity at $\mathbf{r}' = \mathbf{r}$.

The gauge transformation is defined such that it generates the gauge potentials[29]

$$\mathbf{A}_\alpha^{\mathrm{CS}}(\mathbf{r}) = -\frac{\hbar}{e}\nabla_{\mathbf{r}} \int \mathrm{d}^2\mathbf{r}'\theta(\mathbf{r}-\mathbf{r}') \sum_\beta K_{\alpha\beta}\,\rho_\beta(\mathbf{r}'), \tag{2.2}$$

and such that the one-particle Hamiltonian $[-i\hbar\nabla + e\,\mathbf{A}(\mathbf{r})]^2/2m$ for the component α is transformed to

$$H_\alpha = \frac{1}{2m}\left[-i\hbar\nabla + e\,\mathbf{A}(\mathbf{r}) + e\,\mathbf{A}_\alpha^{\mathrm{CS}}(\mathbf{r})\right]^2.$$

Here, m is the mass of the particles, and e is the electron charge. By using $\nabla \times \nabla\theta(\mathbf{r}) = 2\pi\,\delta(\mathbf{r})$, we derive the corresponding magnetic fields,

$$\mathbf{B}_\alpha^{\mathrm{CS}}(\mathbf{r}) = -\frac{h}{e}\sum_\beta K_{\alpha\beta}\,\rho_\beta(\mathbf{r})\mathbf{e}_z.$$

Since $\mathbf{A}_\alpha^{\mathrm{CS}}$ is a gauge field, its Fourier transform $\mathbf{A}_\alpha^{\mathrm{CS}}(\mathbf{q})$ may be fixed to a convenient gauge. We choose it to be transverse, $i\mathbf{q}\cdot\mathbf{A}_\alpha^{\mathrm{CS}}(\mathbf{q}) = 0$, so that it fixes the direction of $\mathbf{A}_\alpha^{\mathrm{CS}}(\mathbf{q})$ to be $\mathbf{e}_z \times \mathbf{q}/|\mathbf{q}|$, up to a sign. For the magnitude, we use that under a Fourier transform $\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r})$ transforms to $\mathbf{B}(\mathbf{q}) = i\mathbf{q} \times \mathbf{A}(\mathbf{q})$, such that we obtain

$$\mathbf{A}_\alpha^{\mathrm{CS}}(\mathbf{q}) = A_\alpha^{\mathrm{CS}}(\mathbf{q})\mathbf{e}_{\mathbf{q}}^\perp = -\frac{h}{e|\mathbf{q}|}\sum_\beta K_{\alpha\beta}\,\rho_\beta(\mathbf{q})\mathbf{e}_{\mathbf{q}}^\perp, \tag{2.3}$$

where we define the transverse unit vector as $\mathbf{e}_{\mathbf{q}}^\perp = i\mathbf{e}_z \times \mathbf{q}/|\mathbf{q}|$.

The effective magnetic field seen by the composite particles of type α is

$$\mathbf{B}_\alpha^* = \mathbf{B} + \langle\mathbf{B}_\alpha^{\mathrm{CS}}\rangle = B\Bigl(1 - \sum_\beta K_{\alpha\beta}\nu_\beta\Bigr)\mathbf{e}_z, \tag{2.4}$$

where ν_β are the component filling factors, given by $\nu_\beta = (h/eB)n_\beta = 2\pi l_B^2 n_\beta$ in terms of the electronic densities n_β and of the magnetic length $l_B = \sqrt{\hbar/eB}$. This result is an extension of the two-component case presented in Refs. 11,12. Notice that each particle type has its own effective magnetic field, and hence also its own magnetic length $l_{B_\alpha^*} = \sqrt{\hbar/eB_\alpha^*}$. The composite particle filling factors ν_α^* are expressed in terms of the electronic filling factors ν_α as[11]

$$\frac{\nu_\alpha^*}{\nu_\alpha} = \frac{l_{B_\alpha^*}^2}{l_B^2} = \frac{B}{B_\alpha^*} = \frac{1}{1 - \sum_\beta K_{\alpha\beta}\,\nu_\beta}. \tag{2.5}$$

This result generalises the one-component relation

$$\nu^* = \frac{\nu}{1-2s\nu} \qquad \leftrightarrow \qquad \nu = \frac{\nu^*}{2s\nu^*+1}, \tag{2.6}$$

in terms of the Chern–Simons charge $K = 2s$.

The statistical angle associated with the exchange of the transformed fields $\psi_\alpha^{\mathrm{CS}}$ and $\psi_\alpha^{\mathrm{CS}\,\dagger}$ can be derived by using their definition, Eq. (2.1), and the fact that the original fields are fermionic. Under the condition that the charge matrix $K_{\alpha\beta}$ is symmetric, which is a generalisation of the condition discussed in the two-component case,[11] we obtain

$$\psi_\alpha^{\mathrm{CS}}(\mathbf{r}_1)\psi_\beta^{\mathrm{CS}}(\mathbf{r}_2) + e^{i\pi\,K_{\alpha\beta}}\psi_\beta^{\mathrm{CS}}(\mathbf{r}_2)\psi_\alpha^{\mathrm{CS}}(\mathbf{r}_1) = 0$$

and

$$\psi_\alpha^{\mathrm{CS}}(\mathbf{r}_1)\psi_\beta^{\mathrm{CS}\,\dagger}(\mathbf{r}_2) + e^{i\pi\,K_{\alpha\beta}}\psi_\beta^{\mathrm{CS}\,\dagger}(\mathbf{r}_2)\psi_\alpha^{\mathrm{CS}}(\mathbf{r}_1) = \delta_{\alpha\beta}\,\delta(\mathbf{r}_1-\mathbf{r}_2).$$

Thus, we have found that the statistical angles of the exchange are $\pi K_{\alpha\beta}$, i.e., proportional to the entries of the charge matrix. The parity of the diagonal elements $K_{\alpha\alpha}$ of the charge matrix K determines the statistical properties of the Chern–Simons fields $\psi_\alpha^{\rm CS}$. If they are even integers, the originally fermionic electron fields ψ_α are transformed into fermionic Chern–Simons fields. However, one may also change the statistical properties of the fields from fermions to bosons by using odd integers for the diagonal components $K_{\alpha\alpha}$. In the following sections, we mainly discuss fermionic Chern–Simons fields, in order to make a connection with the composite-fermion theory, although the main conclusions of the paper also apply to bosonic fields.

III. GAUSSIAN THEORY

A. Auxiliary gauge fields

The formalism proposed by Shankar and Murthy[15,18] allows us to treat the fluctuations of the Chern–Simons vector potential via the introduction of κ real-valued transverse gauge fields $\mathbf{a}_\alpha^\circ(\mathbf{r}_{j_\alpha})$.[24] The extended Chern–Simons Hamiltonian in first quantisation, with N_α particles of each type α, reads

$$H_{CS} = \frac{1}{2m}\sum_\alpha \sum_{j_\alpha=1}^{N_\alpha} \big[\mathbf{p}_{j_\alpha} + e\,\mathbf{A}_\alpha^*(\mathbf{r}_{j_\alpha}) + e\,\delta\mathbf{A}_\alpha^{\rm CS}(\mathbf{r}_{j_\alpha}) + e\,\mathbf{a}_\alpha^\circ(\mathbf{r}_{j_\alpha})\big]^2, \quad (3.1)$$

where we absorb the average value of the Chern–Simons potential (2.2) into an effective vector potential $\mathbf{A}_\alpha^*(\mathbf{r}) = \mathbf{A}(\mathbf{r}) + \langle\mathbf{A}_\alpha^{\rm CS}\rangle$. This definition yields the effective magnetic field $\nabla\times\mathbf{A}_\alpha^*(\mathbf{r}_{j_\alpha}) = \mathbf{B}_\alpha^*(\mathbf{r}_{j_\alpha})$ given in Eq. (2.4). In Fourier space, the fluctuations $\delta\mathbf{A}_\alpha^{\rm CS}(\mathbf{q})$ are transverse, similar to the gauge field itself, as given by Eq. (2.3). Here, we have

$$\delta\mathbf{A}_\alpha^{\rm CS}(\mathbf{q}) = \delta A_\alpha^{\rm CS}(\mathbf{q})\mathbf{e}_\mathbf{q}^\perp = \frac{h}{e|\mathbf{q}|}\sum_\beta K_{\alpha\beta}\,\delta\rho_\beta(\mathbf{q})\mathbf{e}_\mathbf{q}^\perp.$$

Since we have artificially added the auxiliary gauge field $\mathbf{a}_\alpha^\circ(\mathbf{r})$, we have enlarged the Hilbert space, where the physical states form only a subspace $\{|\phi_{\rm phys}\rangle\}$ characterised by

$$a_\alpha^\circ(\mathbf{q})\,|\phi_{\rm phys}\rangle = 0, \quad (3.2)$$

for all components α. In other words, the gauge field operator acting on any physical state vanishes.

Additionally, we introduce a longitudinal field $\mathbf{P}^\circ(\mathbf{q}) = iP^\circ(\mathbf{q})\mathbf{e}_\mathbf{q}^\parallel$ (with $\mathbf{e}_\mathbf{q}^\parallel \equiv \mathbf{q}/|\mathbf{q}|$), conjugate and perpendicular to the newly introduced gauge field $\mathbf{a}^\circ(\mathbf{q}) = a^\circ(\mathbf{q})\mathbf{e}_\mathbf{q}^\perp$, according to the commutation relation in Fourier space

$$[a_\alpha^\circ(\mathbf{q}), P_\beta^\circ(-\mathbf{q}')] = i\hbar\delta_{\alpha\beta}\delta_{\mathbf{q},\mathbf{q}'}.$$

Since the operator P_α° is conjugate to a_α°, it generates translations in a_α°, as may be seen from the definition

$$U = \exp\!\left(\frac{i}{\hbar}\sum_\alpha\sum_{\mathbf{q}'} P_\alpha^\circ(-\mathbf{q}')\delta A_\alpha^{\rm CS}(\mathbf{q}')\right),$$

which translates a_β° by the vector $-\delta A_\beta^{\rm CS}(\mathbf{q})$ as $U^\dagger a_\beta^\circ(\mathbf{q})U = a_\beta^\circ(\mathbf{q}) - \delta A_\beta^{\rm CS}(\mathbf{q})$. By using this shifting property of U, which is also valid in $\mathbf{r}$-space, and with $[\mathbf{p}_{j_\alpha}, U] = (h/eL^2)\sum_\beta K_{\alpha\beta}\mathbf{P}_\beta^\circ(\mathbf{r}_{j_\alpha})$, we may eliminate $\delta\mathbf{A}_\beta^{\rm CS}(\mathbf{q})$ from the Hamiltonian of Eq. (3.1), which then transforms into

$$H_{\rm CP} = U^\dagger H_{\rm CS} U = \frac{1}{2m}\sum_\alpha\sum_{j_\alpha=1}^{N_\alpha}\Big[\mathbf{p}_{j_\alpha} + e\,\mathbf{A}_\alpha^*(\mathbf{r}_{j_\alpha}) + e\,\mathbf{a}_\alpha^\circ(\mathbf{r}_{j_\alpha}) + \frac{h}{eL^2}\sum_\beta K_{\alpha\beta}\mathbf{P}_\beta^\circ(\mathbf{r}_{j_\alpha})\Big]^2,$$

while transforming the states to $\psi^{\rm CP} = U^{-1}\psi^{\rm CS}$. In these equations, L^2 is the area of the system. By transforming the states, we also transform the constraint (3.2) to

$$\big(a_\alpha^\circ(\mathbf{q}) - \delta A_\alpha^{\rm CS}(\mathbf{q})\big)\,|\phi_{\rm phys}\rangle = \left(a_\alpha^\circ(\mathbf{q}) - \frac{2\pi\hbar}{e|\mathbf{q}|}\sum_\beta K_{\alpha\beta}\,\delta\rho_\beta(\mathbf{q})\right)|\phi_{\rm phys}\rangle = 0. \quad (3.3)$$

The Hamiltonian may be decomposed into three terms, $H_{\rm CP} = H^* + H_{\rm coupl} + H_{\rm aux}$, given by

$$H^* = \frac{1}{2m}\sum_\alpha\sum_{j_\alpha=1}^{N_\alpha}\mathbf{\Pi}_{j_\alpha}^2, \quad (3.4)$$

$$H_{\rm coupl} = \frac{1}{m}\sum_\alpha\sum_{j_\alpha=1}^{N_\alpha}\mathbf{\Pi}_{j_\alpha}\cdot\left[e\mathbf{a}_\alpha^\circ(\mathbf{r}_{j_\alpha}) + b\sum_\beta K_{\alpha\beta}\mathbf{P}_\beta^\circ(\mathbf{r}_{j_\alpha})\right], \quad (3.5)$$

$$H_{\rm aux} = \frac{1}{2m}\sum_\alpha\sum_{j_\alpha=1}^{N_\alpha}\left[e^2 {a_\alpha^\circ}^2(\mathbf{r}_{j_\alpha}) + b^2\sum_\beta\sum_\gamma P_\beta^\circ(\mathbf{r}_{j_\alpha})K_{\beta\alpha}K_{\alpha\gamma}P_\gamma^\circ(\mathbf{r}_{j_\alpha})\right], \quad (3.6)$$

where $\mathbf{\Pi}_{j_\alpha} \equiv \mathbf{p}_{j_\alpha} + e\mathbf{A}_\alpha^*(\mathbf{r}_{j_\alpha})$ and $b = h/eL^2$, which has the dimensions of a magnetic field. Notice that for $H_{\rm aux}$ we have used that $\mathbf{a}_\alpha^\circ(\mathbf{q})$ and $\mathbf{P}_\beta^\circ(\mathbf{q})$ are perpendicular.

In the remainder of this paper, we discuss only the term $H_{\rm aux}$ that involves the auxiliary gauge fields. The full theory, including the other terms of the Hamiltonian shall be discussed in a future publication.[25]

B. Gaussian model of the auxiliary gauge fields

We shall now analyse $H_{\rm aux}$ in detail. By observing that $\sum_{j_\alpha}\delta(\mathbf{r}-\mathbf{r}_{j_\alpha}) = \rho_\alpha(\mathbf{r}) = n_\alpha + \delta\rho_\alpha(\mathbf{r})$, we can rewrite Eq. (3.6) as

$$H_{\rm aux} = \frac{1}{2m}\sum_\alpha\int {\rm d}^2\mathbf{r}\,\rho_\alpha(\mathbf{r})\left(e^2 {a_\alpha^\circ}^2(\mathbf{r}) + b^2\sum_\beta\sum_\gamma P_\beta^\circ(\mathbf{r})K_{\beta\alpha}K_{\alpha\gamma}P_\gamma^\circ(\mathbf{r})\right).$$

Up to this point, all equations are exact. Now, we approximate $H_{\rm aux}$ by assuming that the density fluctuations $\delta\rho_\alpha$ are small with respect to the average densities n_α. Since the resulting Hamiltonian becomes quadratic, this approximation is called the *harmonic approximation*. We should keep in mind that this approximation breaks down if the fluctuations are not small with respect to the average densities. In particular, the approximation is certainly invalid if one of the average densities is zero. We therefore assume that none of the average densities n_α vanishes. However, in the case of a singular charge matrix K, a redefinition of the filling factors might lift this problem, as will be discussed in more detail in Sec. IV. The Hamiltonian $H_{\rm aux}$ in Fourier space is approximated by

$$H_{\rm osc} = \sum_{\mathbf{q}} \sum_{\alpha} \frac{n_\alpha L^2}{2m} \Bigg(e^2 a^\circ_\alpha(-\mathbf{q}) a^\circ_\alpha(\mathbf{q}) + b^2 \sum_\beta \sum_\gamma P^\circ_\beta(-\mathbf{q}) K_{\beta\alpha} K_{\alpha\gamma} P^\circ_\gamma(\mathbf{q}) \Bigg), \quad (3.7)$$

where we note that $a^\circ(-\mathbf{q}) = (a^\circ(\mathbf{q}))^\dagger$ and $P^\circ(-\mathbf{q}) = (P^\circ(\mathbf{q}))^\dagger$. Because the Hamiltonian (3.7) is quadratic in the gauge fields a°_α and its conjugate fields P°_α, it is possible to write it in terms of ladder operators. However, due to the appearance of the matrices K in the term with P°'s, it is a nontrivial task to define suitable ladder operators $\mathcal{A}_\alpha(\mathbf{q})$ such that the commutators between them are of the form $[\mathcal{A}_\alpha(\mathbf{q}), \mathcal{A}^\dagger_\beta(\mathbf{q}')] = \delta_{\alpha\beta}\delta_{\mathbf{q},\mathbf{q}'}$.

In order to diagonalise the Hamiltonian, we define $N = \mathrm{diag}(\{\nu_\alpha\})$ as the dimensionless diagonal matrix of filling factors, $N_{\alpha\beta} = \nu_\alpha \delta_{\alpha\beta}$, and also write the fields and their conjugates as vectors in the component space, $a^\circ = (a^\circ_1, \ldots, a^\circ_\kappa)$ and $P^\circ = (P^\circ_1, \ldots, P^\circ_\kappa)$. We omit the $\mathbf{q}$ dependence for a while. In this concise notation, the oscillator Hamiltonian can be written as

$$H_{\rm osc} = \frac{L^2}{2m}\frac{eB}{h}\left[e^2 a^{\circ\dagger} N\, a^\circ + b^2 P^{\circ\dagger}\, K^\dagger N K\, P^\circ\right]. \quad (3.8)$$

The prefactor can also be written as $L^2\omega_{\rm c}/2h$, where $\omega_{\rm c} = eB/m$ is the cyclotron frequency. We recall that the matrix K is real and symmetric, so that $K^\dagger = K$. We perform the diagonalisation in two steps. First, we define $a' = \sqrt{N}a^\circ$ and $P' = \sqrt{N^{-1}}P^\circ$, so that the Hamiltonian becomes

$$H_{\rm osc} = \frac{L^2\omega_{\rm c}}{2h}\left[e^2 a'^\dagger a' + b^2 P'^\dagger \sqrt{N}\, K N K \sqrt{N} P'\right].$$

The matrix between the P''s is the square of the matrix $E \equiv \sqrt{N}\, K \sqrt{N}$, which is real and symmetric. Therefore, it can be diagonalised in terms of a diagonal matrix D and an orthogonal matrix C, such that $E = C^{-1} D\, C$. The matrix D has the eigenvalues λ_α of E on its diagonal, and $C^{\rm T}$ contains the corresponding eigenvectors as columns. The ability to choose C as an orthogonal matrix (i.e., $C^{-1} = C^{\rm T}$) is provided by the property that the matrix E is symmetric, so that the eigenvectors can be chosen such that they form an orthonormal basis. Having found the diagonalisation $E = C^{\rm T} D\, C$, we define

$$\overline{a} = C\, a' = C\sqrt{N} a^\circ, \quad \overline{P} = C\, P' = C\sqrt{N^{-1}} P^\circ, \quad (3.9)$$

so that the Hamiltonian becomes

$$H_{\rm osc} = \frac{L^2\omega_{\rm c}}{2h}\left[e^2 \overline{a}^\dagger \overline{a} + b^2 \overline{P}^\dagger D^2\, \overline{P}\right] \quad (3.10a)$$

$$= \frac{L^2\omega_{\rm c}}{2h} \sum_\alpha \left[e^2 \overline{a}^\dagger_\alpha \overline{a}_\alpha + b^2 \overline{P}^\dagger_\alpha \lambda_\alpha^2\, \overline{P}_\alpha\right], \quad (3.10b)$$

written in matrix form and in components, respectively. For the derivation we have used that $\sum_\alpha \overline{a}^\dagger_\alpha \overline{a}_\alpha = \overline{a}^\dagger \overline{a} = a'^\dagger a'$ by virtue of the orthogonality of C, $\sum_\gamma C_{\alpha\gamma} C_{\beta\gamma} = \sum_\gamma C_{\alpha\gamma} C^{\rm T}_{\gamma\beta} = \delta_{\alpha\beta}$. For this transformation to be well-defined, it is required that $\nu_\beta \neq 0$ for all components β, which we already assumed in order for the harmonic approximation to be valid. The definition is such that the commutator between $\overline{a}$ and $\overline{P}$ is given by

$$[\overline{a}_\alpha(\mathbf{q}), \overline{P}_\beta(-\mathbf{q}')] = i\hbar\, \delta_{\alpha\beta}\delta_{\mathbf{q},\mathbf{q}'}, \quad (3.11)$$

which holds also by virtue of the orthogonality of C.

By setting $P^\circ_\alpha = -i\hbar \frac{\partial}{\partial a^\circ_\alpha}$ and consequently $\overline{P}_\alpha = -i\hbar \frac{\partial}{\partial \overline{a}_\alpha}$, we can derive that

$$\chi_{\rm osc} = \exp\left(-\frac{e}{2\hbar b} \sum_{\mathbf{q}} \sum_\alpha \overline{a}_\alpha(-\mathbf{q}) \xi_\alpha \overline{a}_\alpha(\mathbf{q})\right) \quad (3.12)$$

is a ground state of the Hamiltonian (3.10) if we set $\xi_\alpha = |\lambda_\alpha|^{-1}$. Evidently, ξ_α is only well defined if the matrix E is nonsingular, i.e., if none of its eigenvalues is zero. Moreover, the eigenvalues appearing in the eigenstate are actually not the eigenvalues of E itself, but the square roots of the eigenvalues of $E^2 = \sqrt{N}\, K N K \sqrt{N}$, namely $\sqrt{\lambda_\alpha^2} = |\lambda_\alpha|$. The ground state (3.12) can then be written in matrix form as

$$\chi_{\rm osc} = \exp\left(-\frac{e}{2\hbar b}\overline{a}^\dagger D^{-1} \overline{a}\right) = \exp\left(-\frac{e}{2\hbar b} a^{\circ\dagger} K^{-1} a^\circ\right) \quad (3.13)$$

where we used $\overline{a}^\dagger D^{-1} \overline{a} = a^{\circ\dagger} K^{-1} a^\circ$ in order to write the ground state in terms of the original auxiliary gauge fields a°. Notice that, had we chosen the negative square roots $-\sqrt{\lambda_\alpha^2}$ for the eigenvalues of E, the ground-state wave function (3.13) could not be normalised. Negative eigenvalues are indeed unphysical because they would lead to an instability of the electron liquid, the components of which phase-separate, as may be seen within the plasma picture of the FQHE.[26] It is therefore important, for the structure of the Chern–Simons theory to be well-defined, to discard negative eigenvalues λ_α. This is namely the case for the analysis presented in Sec. III C, where we assume a positive definite K. The case of zero eigenvalues is treated separately in Sec. IV.

Acting with the Hamiltonian (3.10) on the ground state (3.12) gives its energy eigenvalues

$$\sum_\alpha \frac{L^2\omega_{\rm c}}{2h} \hbar e b |\lambda_\alpha| = \frac{\hbar\omega_{\rm c}}{2} \sum_\alpha |\lambda_\alpha| = \frac{\hbar}{2} \sum_\alpha \omega_\alpha,$$

where $\omega_\alpha = |\lambda_\alpha|\omega_{\rm c}$ are the characteristic frequencies, given in terms of the eigenvalues λ_α and the cyclotron frequency $\omega_{\rm c}$.

At this point, we define the ladder operators as

$$\begin{aligned}\mathcal{A}_\alpha(\mathbf{q}) &= \frac{L}{\sqrt{4\pi\hbar^2\lambda_\alpha}}\left(e\,\overline{a}_\alpha(\mathbf{q}) + ib\lambda_\alpha\overline{P}_\alpha(\mathbf{q})\right),\\ \mathcal{A}^\dagger_\alpha(\mathbf{q}) &= \frac{L}{\sqrt{4\pi\hbar^2\lambda_\alpha}}\left(e\,\overline{a}_\alpha(-\mathbf{q}) - ib\lambda_\alpha\overline{P}_\alpha(-\mathbf{q})\right),\end{aligned} \tag{3.14}$$

still under the assumption that the eigenvalues λ_α are positive. The commutator of the rescaled ladder operators becomes $[\mathcal{A}_\alpha(\mathbf{q}), \mathcal{A}^\dagger_\beta(\mathbf{q}')] = \delta_{\alpha\beta}\delta_{\mathbf{q},\mathbf{q}'}$, so that $\mathcal{A}^\dagger_\alpha(\mathbf{q})\mathcal{A}_\alpha(\mathbf{q})$ is the number operator for the oscillator states in the component α of the diagonalised basis. The Hamiltonian can be conveniently written in terms of the ladder operators as

$$H_{\rm osc} = \sum_{\mathbf{q}}\sum_{\alpha}\hbar\omega_\alpha\big(\mathcal{A}^\dagger_\alpha(\mathbf{q})\mathcal{A}_\alpha(\mathbf{q}) + \tfrac{1}{2}\big). \tag{3.15}$$

This result also proves that the "ground state" (3.12) is indeed the lowest-energy state.

Notice that the energies $\hbar\omega_\alpha$ play the role of *quasi-particle gaps* in the Chern–Simons theory, and the ground state is well-defined for $\det(K) \neq 0$.[8] Zero-energy gaps are obtained if one of the eigenvalues $\lambda_\alpha = 0$, i.e., when the matrix K is singular, $\det(K) = \det(E) = 0$. Contrary to what one may naively expect, this situation is not in contradiction with an incompressible quantum liquid, where all (collective) charge modes must be gapped. As we discuss in more detail in Sec. IV, the zero-gap modes associated with $\lambda_\alpha = 0$ reveal ferromagnetic properties of the underlying state,[22] which in the presence of interactions evolve into spin-wave modes while keeping the charge modes gapped.

C. Connection with trial wave functions

In order to obtain the wave functions corresponding to the ground state (3.13), we may rewrite it in terms of the density fluctuations $\delta\rho_\alpha(\mathbf{q})$, using the constraint (3.3). Once again, it is more convenient to do the computation in matrix notation. The constraint is then given by $a^\circ = (h/e|\mathbf{q}|)K\,(\delta\rho)$ for physical states, with $(\delta\rho) = (\delta\rho_1, \ldots, \delta\rho_\kappa)$ the vector of the density fluctuations. Hence, we find

$$\chi_{\rm osc} = \exp\!\left(-\frac{1}{2}(\delta\rho)^\dagger\frac{2\pi L^2}{|\mathbf{q}|^2}K\,(\delta\rho)\right). \tag{3.16}$$

Notice that, written in terms of density fluctuations, the ground-state wave function is no longer confronted with the problem of zero-eigenvalues of E (or K) because it is the matrix K, and not its inverse K^{-1}, which appears here.

As shown in Ref. 21, we may relate the expression (3.16) to the plasma picture proposed by Laughlin in his original publication.[1] In this picture, we regard $|\chi_{\rm osc}|^2$ as the Boltzmann weight $\exp(-\beta\mathcal{H})$ of the plasma Hamiltonian $\mathcal{H}$, where one sets $\beta = 2$ (Ref. 26). Then $\mathcal{H}$ can be identified as the Hamiltonian of particles interacting due to the Coulomb potential in two dimensions, $-\log|\mathbf{r}|$, which equals $2\pi L^2/|\mathbf{q}|^2$ in momentum space. As discussed in Appendix A, the wave function that we obtain is

$$\psi(\{z_{j_\alpha}\}) = \prod_\alpha\prod_{\substack{j_\alpha,k_\alpha\\ j_\alpha<k_\alpha}}(z_{j_\alpha}-z_{k_\alpha})^{K_{\alpha\alpha}}\prod_{\substack{\alpha,\beta\\ \alpha<\beta}}\prod_{j_\alpha,k_\beta}(z_{j_\alpha}-z_{k_\beta})^{K_{\alpha\beta}}\exp\!\left(-\sum_{\alpha,\beta}\nu_\alpha K_{\alpha\beta}\sum_{k_\beta}\frac{|z_{k_\beta}|^2}{4l_B^2}\right)\phi_{\{\nu^*_\alpha\}}(\{z_{j_\alpha}\}), \tag{3.17}$$

where we write $z = x - iy$. This wave function is a product of the oscillator function and the wave function $\phi_{\{\nu^*_\alpha\}}(\{z_{j_\alpha}\})$, which encodes the residual degrees of freedom for particles in the reduced field $\mathbf{B}_\alpha$, i.e., at the effective filling factors ν^*_α given by Eq. (2.5). Quite generally, one may describe the same system in the framework of different Chern–Simons theories, according to how much flux is absorbed in the transformation by the matrix $K_{\alpha\beta}$. It is often convenient, if possible, to choose the Chern–Simons transformation such that the residual wave function is factorisable into single-component wave functions $\tilde{\phi}_{\nu^*_\alpha}$,

$$\phi_{\{\nu^*_\alpha\}}(\{z_{j_\alpha}\}) = \prod_{\alpha=1}^{\kappa}\tilde{\phi}_{\nu^*_\alpha}(\{z_{j_\alpha}\}), \tag{3.18}$$

so that each component may be treated independently after the transformation. Notice, however, that this aim may be in conflict with the above-mentioned condition of positive eigenvalues of the charge matrix $K_{\alpha\beta}$, namely in the context of symmetric states with ferromagnetic properties that we discuss in Sec. IV B.

The simplest state of a factorisable residual wave function according to Eq. (3.18) consists of a product of states at an effective filling factor $\nu^*_\alpha = 1$ for each component, each of which involves a Slater determinant, in the form

$$\tilde{\phi}_{\nu^*_\alpha=1}(\{z_{j_\alpha}\}) = \prod_{j_\alpha<k_\alpha}(z_{j_\alpha}-z_{k_\alpha})\exp\!\left(-\sum_{k_\alpha}\frac{|z_{k_\alpha}|^2}{4l^2_{B^*_\alpha}}\right). \tag{3.19}$$

Such a state would then correspond to a Halperin wave function that is described by an exponent matrix $M_{\alpha\beta} = K_{\alpha\beta} + \delta_{\alpha\beta}$. In order to have a fermionic wave function, the elements $K_{\alpha\alpha}$ must naturally be even integers, and we thus have a Chern–Simons theory that transforms fermions into (compos-

ite) fermions. Alternatively, one may have chosen the bosonic version of the Chern–Simons theory, in which case the diagonal elements of the matrix $K_{\alpha\beta} = M_{\alpha\beta}$ would be odd. The same state (3.17) would then be described as a product of the oscillator wave function $\chi_{\rm osc}$, which absorbs all the flux, and a *bosonic* wave function for zero net magnetic field $B^*_\alpha = 0$, for all components, $\phi_{\{B^*_\alpha=0\}}(\{z_{j_\alpha}\}) = 1$.

Until now, we have discussed states that may be described in terms of generalised κ-component Halperin wave functions,[22] where the residual wave function $\phi_{\{\nu^*_\alpha\}}(\{z_{j_\alpha}\})$ is itself a (typically simpler) Halperin wave function described by a "residual" exponent matrix $M^*_{\alpha\beta}$ such that $M_{\alpha\beta} = K_{\alpha\beta} + M^*_{\alpha\beta}$ (see also Appendix A). Notice, however, that the Chern–Simons theory discussed above may also provide us with another class of factorisable trial wave functions if we replace the Slater determinants (3.19) for the effective filling factors $\nu^*_\alpha = 1$ by Slater determinants for p_α completely filled composite-fermion levels $\phi^{(\alpha)}_{p_\alpha}(\{z_{j_\alpha}\})$ in each component. The resulting wave function (3.17) is related to the κ-component Halperin wave function in the same manner as Jain's one-component composite-fermion[2,3] to Laughlin's wave function.[1] Naturally, the proposed Slater determinants contain non-analytic components in the polynomial, and, in the same manner as for Jain's wave functions, one needs to project the resulting wave function to the subspace of analytic functions in order to satisfy the lowest-Landau-level condition.

Ultimately, the theory may be generalised to the case where the ν^*_α's can take any fractional value, as to allow the multi-component generalisation of higher-generation FQHE states. An example of the latter in one component is the $\nu = 4/11$ state, which can be understood as a second generation FQHE state.[19,27]

IV. SINGULAR TRANSFORMATIONS

The analysis in the previous section demonstrates that a Chern–Simons transformation with a nonsingular charge matrix is already interesting in itself. However, transformations with singular charge matrices play an important role in the study of states with (partial) ferromagnetic order, since these states are described by singular exponent matrices.[22] In this section, we investigate the consequences of the symmetry properties of the exponent matrices M and M^* and the charge matrix K for the results of the previous section.

A. Conditions on the ranks of the matrices

Without performing the diagonalisation of the oscillator Hamiltonian, it is already possible to give some conditions on the exponent matrices and the charge matrix. Consider a state that is described by a singular exponent matrix M. As a consequence, not all filling factors are defined separately. Suppose furthermore that the electronic and composite-fermion filling factors are given by $\sum_\beta M_{\alpha\beta}\nu_\beta = 1$ and $\sum_\beta M^*_{\alpha\beta}\nu^*_\beta = 1$, respectively, with $M = M^* + K$. We note that Eq. (2.5) has to be satisfied simultaneously, which does not necessarily follow from the other conditions.[30] From the fact that M, M^* and K are required to be nonnegative definite, it follows that also K and M^* are singular. More specifically, it follows that the null spaces of M^* and K may be of higher dimension than that of M. As a consequence, the dimension of the null space of the exponent matrix is either increased or kept invariant by the Chern–Simons transformation. In other words, if before applying the Chern–Simons transformation the theory involves a certain number of independent combinations of filling factors, then the number of independent combinations after the transformation is either the same or lower. In terms of the ranks of the matrices, which is equal to their size minus the dimension of the null space (i.e., $\dim\ker M + \operatorname{rank} M = \kappa$), we find that the ranks of K and M^* must both be smaller than or equal to the rank of M.

For the case that $\operatorname{rank} M^* < \operatorname{rank} M$, which is not ruled out by the above discussion, some problems may arise. In this case, Eq. (2.5) fixes the filling factors ν^*_α to be confined to a subspace of the space of all solutions of $\sum_\beta M^*_{\alpha\beta}\nu^*_\beta = 1$. For example, if $M = \left(\begin{smallmatrix} 3 & 1 \\ 1 & 3 \end{smallmatrix}\right)$ and $K = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right)$, we have $M^* = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, so that, based on the exponent matrices, the electronic and composite-fermion filling factors are given by $(\nu_1, \nu_2) = (1/4, 1/4)$ and $\nu^*_1 + \nu^*_2 = 1$, respectively. However, based on Eq. (2.5), the composite-fermion filling factors are fixed at $(\nu^*_1, \nu^*_2) = (1/2, 1/2)$. Therefore, the matrix M^* does not appropriately describe the possible composite-fermion filling factors of the system. We would expect that this leads to problematic results, if we used the Chern–Simons approach to obtain a separation between high-energy and low-energy degrees of freedom. For this reason, we will only analyse the case that M and M^* share their ranks. We stress that there is no problem in using a singular charge matrix K if M and M^* are both nonsingular.

B. The oscillator Hamiltonian

Here, we discuss how the singularity of the matrix K affects the analysis that we used to study the harmonic oscillator. Apart from the zero modes in the harmonic oscillator, we must also take into account that the number of independent constraints [Eq. (3.3)] is reduced, since these also involve the matrix K. Indeed, the number of independent constraints is given by the rank r of the matrix K, whereas the number of zero modes is $\kappa - r$. Before we derive the fully general results, we find it instructive to illustrate the procedure first with a simple example.

We consider a two-component system, where we choose the charge matrix of the Chern–Simons transformation to be the singular matrix $K = \left(\begin{smallmatrix} 2 & 2 \\ 2 & 2 \end{smallmatrix}\right)$. The eigenvalues of K are 4 and 0, and the respective eigenvectors are $(1,1)/\sqrt{2}$ and $(1,-1)/\sqrt{2}$. We can write the constraints in components as

$$0 = \left(a^\circ_\alpha(\mathbf{q}) - \frac{h}{e|\mathbf{q}|}\big(2\,\delta\rho_1(\mathbf{q}) + 2\,\delta\rho_2(\mathbf{q})\big)\right)|\phi_{\rm phys}\rangle\,,$$

for $\alpha = 1, 2$. The two components a°_1 and a°_2 of the gauge

field satisfy the same constraint, so that they are fixed to the fluctuations of the total density $\delta\rho_1+\delta\rho_2$. On the other hand, the difference of density fluctuations $\delta\rho_1-\delta\rho_2$ (associated with the zero eigenvalue) is absent, implying that one may have zero-energy fluctuations that lower the particle number in one component while increasing that in the other component. Eventually such a reorganisation of the particles on the two components may even completely polarise the system, with $\nu_1=\nu$ and $\nu_2=0$. Inversely this means that in the case of a singular matrix K, we may always choose both filling factors nonzero or even equal, i.e., N nonsingular, such as to render the harmonic approximation (3.7) valid.

We now turn to the harmonic oscillator Hamiltonian. We write $N=\mathrm{diag}(\nu_1,\nu_2)$, where $\nu_{1,2}$ are the electronic filling factors. For this example, we compute

$$E=\sqrt{N}\,K\sqrt{N}=2\begin{pmatrix}\nu_1 & \sqrt{\nu_1\nu_2}\\ \sqrt{\nu_1\nu_2} & \nu_2\end{pmatrix}.$$

This matrix is diagonalised as $C^{\mathrm{T}}DC$, where $D=\mathrm{diag}(\lambda_1,\lambda_2)$ is the diagonal matrix with the eigenvalues $\lambda_1=2(\nu_1+\nu_2)$ and $\lambda_2=0$. The corresponding eigenvectors are proportional to $(\sqrt{\nu_1},\sqrt{\nu_2})$ and $(-\sqrt{\nu_2},\sqrt{\nu_1})$, respectively. The diagonalised Hamiltonian is given by Eq. (3.10b), where $\alpha=1,2$ and

$$\begin{pmatrix}\overline{a}_1\\ \overline{a}_2\end{pmatrix}=\frac{1}{\sqrt{\nu_1+\nu_2}}\begin{pmatrix}\nu_1 a_1^\circ+\nu_2 a_2^\circ\\ \sqrt{\nu_1\nu_2}\,(-a_1^\circ+a_2^\circ)\end{pmatrix},$$
$$\begin{pmatrix}\overline{P}_1\\ \overline{P}_2\end{pmatrix}=\frac{1}{\sqrt{\nu_1+\nu_2}}\begin{pmatrix}P_1^\circ+P_2^\circ\\ -\sqrt{\frac{\nu_2}{\nu_1}}\,P_1^\circ+\sqrt{\frac{\nu_1}{\nu_2}}P_2^\circ\end{pmatrix}.$$

We note that $\overline{P}_2$ is not present in the Hamiltonian since the term $\overline{P}_2^\dagger\lambda_2^2\overline{P}_2$ vanishes due to $\lambda_2=0$. The term $\overline{a}_2^\dagger\overline{a}_2$ also vanishes, since $\overline{a}_2=0$, due to the constraint $a_1^\circ=a_2^\circ$. In the end, we obtain a harmonic oscillator Hamiltonian with only one coordinate ($\overline{a}_1$) and one momentum ($\overline{P}_1$) component.

The Hamiltonian restricted to this single coordinate has a ground state $\chi_{\mathrm{osc},1}=\exp[-(e/2\hbar b)\overline{a}^\dagger\widehat{D}\,\overline{a}]$, where we define $\widehat{D}=\mathrm{diag}(\frac{1}{2}(\nu_1+\nu_2)^{-1},0)$. We note that $\chi_{\mathrm{osc},1}$ only involves $\overline{a}_1$, but not $\overline{a}_2$. Transforming back to the coordinates (a_1°,a_2°) and imposing the constraints $a_1^\circ=a_2^\circ=(h/e|\mathbf{q}|)\big(2\,\delta\rho_1+2\,\delta\rho_2\big)$, we obtain

$$\chi_{\mathrm{osc},1}=\exp\!\left(-\frac{1}{2}(\delta\rho_1+\delta\rho_2)^\dagger\frac{2\pi L^2}{|\mathbf{q}|^2}(2)\,(\delta\rho_1+\delta\rho_2)\right),$$

where the notation (2) is to point out that it should be interpreted as a matrix. At this point, we observe that $(\delta\rho_1+\delta\rho_2)^\dagger(2)(\delta\rho_1+\delta\rho_2)$ is exactly equal to $(\delta\rho_1,\delta\rho_2)^\dagger K(\delta\rho_1,\delta\rho_2)$. This means that in this example Eq. (3.16) is valid without change, and the other results concerning the Halperin wave functions hold as well, as we have already mentioned in the discussion of the general oscillator function (3.16). We remark that the linear combination of filling factors $\nu_1-\nu_2$ is not present at all in the diagonalised theory.

Another important point is that we can make the connection with ferromagnetic Laughlin states in two-component systems.[26] For instance, the exponent matrix $M=\left(\begin{smallmatrix}3&3\\3&3\end{smallmatrix}\right)$ defines a state for which the total filling factor is $\nu_1+\nu_2=1/3$, but the separate filling factors are not defined, since the exponent matrix is singular.[26] Using the Chern–Simons transformation of the example above, we may understand this state in terms of a composite-fermion theory with exponent matrix $M^*=\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)$. This state has total composite-fermion filling factor $\nu_1^*+\nu_2^*=1$, and again the separate filling factors are undefined. We remark that although the intermediate steps in the procedure contain the separate filling factors ν_1 and ν_2, the results are completely independent of $\nu_1-\nu_2$.

In contrast to the ferromagnetic Laughlin state discussed in the preceding paragraph, we may also discuss the two-component state at total filling factor $\nu=2/5$, described by the matrix $M=\left(\begin{smallmatrix}3&2\\2&3\end{smallmatrix}\right)$ and the reduced exponent matrix $M^*=\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$. Although the Chern–Simons transformation is described by a singular charge matrix K and does therefore not impose a constraint on the relative particle distribution on the two components, the constraint is imposed by M^*, $\nu_1^*=\nu_2^*=1$. The state thus described is then a spin-unpolarised state, as one could have also expected from the original exponent matrix M.

The reasoning given for the example above can be readily generalised to any situation in which K is singular. Here, we assume that the rank r of the matrix K is smaller than the number of components κ. As argued in Appendix B, the ground state can be decomposed as a product of the usual ground state (3.13) restricted to the r independent components, $\chi_{\mathrm{osc},r}$ and the degenerate part $\tilde{\chi}$ [see Eqs. (B1) and (B2)]. Moreover, Eq. (3.16) remains valid even in the singular case, despite the fact that the original derivation involves the inverse of K. Hence, the Halperin connection in Sec. III C is valid in the singular case without modification.

The equivalence of the decomposition (B1) for the κ-component oscillator wave function may be interpreted in a straightforward physical manner. Indeed, the decomposition indicates that, in the case of a charge matrix K of rank r, the "reduced" r-component wave function corresponds to an r-component Halperin wave function with gapped oscillator frequencies ω_α. The other factor $\tilde{\chi}$ in Eq. (B1) corresponds to the $\kappa-r$ zero eigenvalues of the matrix K with an associated space spanned by the oscillator components $\bar{a}_\alpha$, with $\alpha=r+1,\ldots,\kappa$. The ground-state manifold comprises therefore any possible combination of these components $\bar{a}_\alpha$, and a particular choice spontaneously breaks the residual ground-state symmetry, which may be related to the ferromagnetic properties of the Halperin state, and $\tilde{\chi}$ may then be interpreted as the ferromagnetic part of the wave function.

In order to see this particular point, consider the r constraints to fix the filling factors of the first $r-1$ components. The last constraint then imposes simply the *sum* of the fillings of all other components $\alpha=r,\ldots,\kappa$. This is naturally a simplified assumption, because the r constraints do not in general fix particular components, but the dependencies may be more complicated.[31] One is then free to distribute the involved particles over these components in a quantum mechanical manner. All different distributions define the ground-state manifold. Schematically, this may be formalised with the help

of a wave function

$$\tilde{\chi} = u_r \left|\alpha = r\right\rangle + u_{r+1} \left|\alpha = r+1\right\rangle + \ldots + u_\kappa \left|\alpha = \kappa\right\rangle ,$$

where the complex amplitudes u_α are subject to a normalisation condition, which plays the role of the last constraint. These complex amplitudes may be viewed as the components of a $\mathrm{CP}^{\kappa-r}$ field.[32] The ground-state manifold may then be described by spatially constant $\mathrm{CP}^{\kappa-r}$ fields with a global $\mathrm{SU}(\kappa-r+1)$ symmetry, which is precisely the symmetry group that describes the ferromagnetic properties of the oscillator wave function. In summary, this argument shows that, in the case of a Chern–Simons transformation with a matrix K of rank r, one may decompose an arbitrary oscillator wave function into a product of a reduced r-component Halperin wave function and a $\mathrm{SU}(\kappa-r+1)$-symmetric ferromagnetic part. Naturally, this symmetry may be further reduced if the components of the Chern–Simons field fix further filling factors.

We finally mention that the spontaneous breaking of the $\mathrm{SU}(\kappa-r+1)$ symmetry yields Goldstone modes, which are physical (pseudo)spin waves. On the level of the Gaussian model, these Goldstone modes are dispersionless and remain at zero energy. This is no longer the case if one takes into account interactions between the particles associated with the different components. One may indeed treat rather easily a density-density interaction within the present model. This interaction may be translated, via the constraints (3.3), into an interaction between the oscillator fields, which one can then diagonalise within the Gaussian model. Notice that these fields are coupled to the $\mathbf{\Pi}_\alpha$ [see Eq. (3.5)], which describe the low-energy electronic degrees of freedom. A discussion of collective Goldstone-type modes is therefore more involved and requires a decoupling of the oscillator and the electronic degrees of freedom. However, the Chern–Simons analysis within the Gaussian model yields valuable insight into the ferromagnetic properties of the states, which are governed by symmetry, as well as into the number of their Goldstone modes.

V. CONCLUSIONS

In conclusion, we have studied a microscopic Chern–Simons approach to general multi-component quantum Hall systems (with κ components). Beyond the mean-field approximation, which yields a renormalisation of the magnetic field that depends on the average particle densities for each component, their fluctuations are taken into account within a Gaussian model of auxiliary gauge fields. These gauge fields, introduced by Shankar and Murthy in the framework of the Hamiltonian theory of the FQHE,[14–18] are indeed connected via constraints to the component density fluctuations.

The analysis of the Gaussian model —although it may be viewed as a first step in the discussion of a more complete Hamiltonian theory for multi-component quantum Hall systems— already yields valuable insight into the structure and the correctness of the Chern–Simons theory, which is characterised by a symmetric $\kappa \times \kappa$ charge matrix K.[9] Most saliently, one needs to discard charge matrices with negative eigenvalues because the associated Chern–Simons theories yield oscillator ground-state wave functions that are not normalised. This is in line with physical insight obtained from a multi-component version of Laughlin's plasma picture[1] according to which charge matrices with negative eigenvalues yield inhomogeneous ground states where the components phase-separate.[26]

Whereas singular charge matrices, with zero eigenvalues, had originally been discussed by Lopez and Fradkin[10] only for the $\mathrm{SU}(2)$-symmetric case, we have argued here that the associated Chern–Simons theories reflect underlying ferromagnetic states in a more general setting. Indeed, we have shown that the density fluctuations of the κ components are then determined by only $r < \kappa$ constraints, such that $\kappa - r$ particular combinations of the component densities may be chosen freely in the ground-state manifold, which is thus described by the $\mathrm{SU}(\kappa-r+1)$ group. This symmetry is spontaneously broken by a particular ferromagnetic state, which can be described by $\kappa-r$ different Goldstone modes that may be viewed as generalised spin waves. Our results encompass the particular $\mathrm{SU}(2)$ case of two-component Chern–Simons theories discussed in the literature.[8–12]

We emphasise moreover that the analysis of the microscopic multi-component Chern–Simons theory within the Gaussian approximation heuristically yields trial wave functions for multi-component quantum Hall systems that may be further studied numerically. As an example, we have discussed generalised κ-component Halperin wave functions that play a similarly central role as Laughlin's wave functions do in one-component quantum Hall systems. Beyond these generalised Halperin wave functions, we have briefly discussed a second class of states, where the residual wave function that is not encoded in the Chern–Simons oscillator part χ_{osc} is a product of Slater determinants of p_α completely filled (α-component) composite-fermion levels. This construction is reminiscent of Jain's generalisation of one-component Laughlin wave functions to filling factors $\nu = p/(2sp+1)$.[2,3,6]

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Appendix A: Multi-component plasma analogy

The single-component plasma analogy proposed by Laughlin[1] is readily generalised to the multi-component case. Here, we use the ground state

$$\chi_{\mathrm{osc}} = \exp\biggl(-\frac{1}{2}\sum_{\mathbf{q}}\sum_{\beta,\gamma}\delta\rho_\beta(-\mathbf{q})\frac{2\pi L^2}{|\mathbf{q}|^2}K_{\beta\gamma}\delta\rho_\gamma(\mathbf{q})\biggr), \tag{A1}$$

which is Eq. (3.16) written out in components. Recalling that $2\pi L^2/|\mathbf{q}|^2$ is the Fourier transform of $-\log|\mathbf{r}|$, we perform an inverse Fourier transformation and we substitute the density fluctuations $\delta\rho_\alpha(\mathbf{r}) = \sum_{j_\alpha} \delta(\mathbf{r}-\mathbf{r}_{j_\alpha}) - n_\alpha$. Then, we can rewrite $\chi_{\rm osc}$ as

$$\chi_{\rm osc} = \exp\left[\frac{1}{2}\sum_{\alpha,\beta} K_{\alpha\beta}\int \mathrm{d}^2\mathbf{r}\mathrm{d}^2\mathbf{r}'\Big(\sum_{j_\alpha=1}^{N_\alpha}\delta(\mathbf{r}-\mathbf{r}_{j_\alpha})-n_\alpha\Big)\log|\mathbf{r}-\mathbf{r}'|\Big(\sum_{k_\beta=1}^{N_\beta}\delta(\mathbf{r}'-\mathbf{r}_{k_\beta})-n_\beta\Big)\right].$$

By evaluating the integrals, one finds

$$\chi_{\rm osc} = \mathrm{const}\cdot\prod_{\alpha,\beta}\prod_{\substack{j_\alpha,k_\beta\\ j_\alpha\neq k_\beta}}|\mathbf{r}_{j_\alpha}-\mathbf{r}_{k_\beta}|^{K_{\alpha\beta}/2}\exp\Big(-\frac{\pi}{2}\sum_{\alpha,\beta}n_\alpha K_{\alpha\beta}\sum_{k_\beta}|\mathbf{r}_{k_\beta}|^2\Big).$$

Using that $\nu_\alpha = 2\pi l_B^2 n_\alpha$, and changing to complex notation, with $z = x - iy$,[33] we can explicitly write this expression as

$$\chi_{\rm osc} = \mathrm{const}\cdot\prod_{\alpha}\prod_{\substack{j_\alpha,k_\alpha\\ j_\alpha<k_\alpha}}|z_{j_\alpha}-z_{k_\alpha}|^{K_{\alpha\alpha}}\prod_{\substack{\alpha,\beta\\ \alpha<\beta}}\prod_{j_\alpha,k_\beta}|z_{j_\alpha}-z_{k_\beta}|^{K_{\alpha\beta}}\exp\Big(-\sum_{\alpha,\beta}\nu_\alpha K_{\alpha\beta}\sum_{k_\beta}\frac{|z_{k_\beta}|^2}{4l_B^2}\Big).$$

The Jastrow-like products in this expression only contain distances between the particles, i.e., only the moduli $|z_{j_\alpha}-z_{k_\beta}|$. Phase factors of the form $[(z_{j_\alpha}-z_{k_\beta})/|z_{j_\alpha}-z_{k_\beta}|]^{K_{\alpha\beta}} = \exp[iK_{\alpha\beta}\arg(z_{j_\alpha}-z_{k_\beta})] = \exp[-iK_{\alpha\beta}\theta(\mathbf{r}_{j_\alpha}-\mathbf{r}_{k_\beta})]$ are obtained from substitution of the full density $\rho_\alpha(\mathbf{r}) = \sum_{j_\alpha}\delta(\mathbf{r}-\mathbf{r}_{j_\alpha})$ into the Chern–Simons transformation (2.1). Applying this transformation to $\chi_{\rm osc}$, we obtain the product of the latter with the phase factors,

$$\psi(\{z_{j_\alpha}\}) = \prod_{\alpha}\prod_{\substack{j_\alpha,k_\alpha\\ j_\alpha<k_\alpha}}(z_{j_\alpha}-z_{k_\alpha})^{K_{\alpha\alpha}}\prod_{\substack{\alpha,\beta\\ \alpha<\beta}}\prod_{j_\alpha,k_\beta}(z_{j_\alpha}-z_{k_\beta})^{K_{\alpha\beta}}\exp\Big(-\sum_{\alpha,\beta}\nu_\alpha K_{\alpha\beta}\sum_{k_\beta}\frac{|z_{k_\beta}|^2}{4l_B^2}\Big)\,\phi_{\{\nu_\alpha^*\}}(\{z_{j_\alpha}\}), \tag{A2}$$

where $\phi_{\{\nu_\alpha^*\}}$ denotes the composite-particle wave function for filling factors ν_α^*, which will be investigated in the following. The magnetic lengths appearing in $\phi_{\{\nu_\alpha^*\}}(\{z_{j_\alpha}\})$ are the reduced magnetic lengths $l_{B_\alpha^*}$ given by Eq. (2.5).

As an example, we consider the situation in which ν_α^* can be determined by an exponent matrix M^*,[9,22,26] such that $\phi_{\{\nu_\alpha^*\}}$ is the Halperin wave function

$$\phi_{\{\nu_\alpha^*\}}(\{z_{j_\alpha}\}) = \prod_{\alpha}\prod_{\substack{j_\alpha,k_\alpha\\ j_\alpha<k_\alpha}}(z_{j_\alpha}-z_{k_\alpha})^{M^*_{\alpha\alpha}}\prod_{\substack{\alpha,\beta\\ \alpha<\beta}}\prod_{j_\alpha,k_\beta}(z_{j_\alpha}-z_{k_\beta})^{M^*_{\alpha\beta}}\exp\Big(-\sum_{\alpha}\sum_{k_\alpha}\frac{|z_{k_\alpha}|^2}{4l_{B_\alpha^*}^2}\Big). \tag{A3}$$

Combining Eqs. (A2) and (A3), we obtain the full electronic wave function,

$$\psi(\{z_{j_\alpha}\}) = \prod_{\alpha}\prod_{\substack{j_\alpha,k_\alpha\\ j_\alpha<k_\alpha}}(z_{j_\alpha}-z_{k_\alpha})^{K_{\alpha\alpha}+M^*_{\alpha\alpha}}\prod_{\substack{\alpha,\beta\\ \alpha<\beta}}\prod_{j_\alpha,k_\beta}(z_{j_\alpha}-z_{k_\beta})^{K_{\alpha\beta}+M^*_{\alpha\beta}}\exp\Big(-\sum_{\alpha}\sum_{k_\alpha}\frac{|z_{k_\alpha}|^2}{4l_B^2}\Big), \tag{A4}$$

which is the Halperin wave function for the exponent matrix $M_{\alpha\beta} = M^*_{\alpha\beta} + K_{\alpha\beta}$.[22] Here, we have expressed the effective magnetic lengths in the exponential of Eq. (A3) in terms of the original one, as $1/l_{B_\alpha^*}^2 = (1-\sum_\beta K_{\alpha\beta}\nu_\beta)/l_B^2$, by virtue of Eq. (2.5).

Appendix B: Ground state in the singular case

The reasoning given for the two-component example in Sec. IV B can be extended to any number of components. Suppose that the charge matrix K (being a $\kappa\times\kappa$ symmetric non-negative definite matrix) is of rank r, which means that it has r independent rows or columns. In particular, there are $\kappa - r$ rows or columns that can be written as a linear combination of the other r independent rows or columns. This also means

that the dimension of the null space, or equivalently, the multiplicity of zero eigenvalues is equal to $\kappa - r$.

Since the constraints (3.3) are expressed as a linear relation involving the matrix K, there are only r independent constraints. Hence, the vector $a^\circ = (a_1^\circ, \ldots, a_\kappa^\circ)$ lives only in an r-dimensional subspace; $\kappa - r$ of its components can be written as a linear combination of the other r.

Now we analyse the Hamiltonian (3.7). Since we have assumed that the matrix of densities N is nonsingular (i.e., all filling factors are nonzero, as required for the harmonic approximation to be valid), the rank of $E = \sqrt{N}\,K\sqrt{N}$ is equal to the rank of K. This means that E has r positive eigenvalues and $\kappa - r$ zero eigenvalues, just as the matrix K. We diagonalise E as usual in terms of a diagonal matrix D and an orthogonal matrix C such that $E = C^{\mathrm{T}} D\, C$. Note that the order of the eigenvalues on the diagonal of D (and simultaneously the order of the rows of C) may be chosen at will, so that we may choose for simplicity $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_r, 0, \ldots, 0)$, where $\lambda_1, \ldots, \lambda_r$ are the positive eigenvalues of E. In the diagonalised Hamiltonian (3.10), the components $\overline{P}_{r+1}, \ldots, \overline{P}_\kappa$ are absent since they are multiplied with the zero eigenvalues of D. We still have κ components of $\overline{a}$ in the Hamiltonian, but we should remember that only r of them are independent.

The diagonalised Hamiltonian contains r nonzero eigenvalues, which depend on the filling factors ν_α. However, some variations of the filling factors will leave the eigenvalues, and hence the diagonalised Hamiltonian, invariant, namely those satisfying the equation

$$0 = (\nabla\lambda_\beta)\cdot\delta\nu = \sum_\alpha \frac{\partial\lambda_\beta}{\partial\nu_\alpha}\delta\nu_\alpha \quad \text{for all } \beta.$$

In other words, the desired variations are the vectors in the null space of the gradient matrix $(\nabla\lambda)$ of the eigenvalues, which is defined as the matrix of derivatives of λ, with respect to ν, $(\nabla\lambda)_{\beta\alpha} = \partial\lambda_\beta/\partial\nu_\alpha$. Since $\kappa - r$ of the eigenvalues λ_β are zero, the rank of the gradient matrix is at most r, and this consequently means that we can find at least $\kappa - r$ independent variations of the filling factors which leave the eigenvalues invariant.

In the example discussed in Sec. IV B, we observed that $\nu_1 - \nu_2$ does not appear in the diagonalised Hamiltonian. In order to demonstrate the procedure sketched in the preceding paragraph, we compute the gradients of the eigenvalues 0 and $2(\nu_1 + \nu_2)$. Obviously, in this example the gradient matrix is $(\nabla\lambda) = \left(\begin{smallmatrix} 0 & 0 \\ 2 & 2 \end{smallmatrix}\right)$ and its null space is spanned by the single vector $(1, -1)$. Since this vector is independent of the filling factors ν_1 and ν_2, we can state that all eigenvalues, and hence the diagonalised Hamiltonian, are invariant under the transformation $\{\nu_1 \to \nu_1 + \delta\nu, \nu_2 \to \nu_2 - \delta\nu\}$. This means that the linear combination $\nu_1 - \nu_2$ is completely absent from the Hamiltonian, as argued earlier. We remark that the variation that leaves the Hamiltonian invariant need not always be constant in the filling factors ν_α. However, for physically relevant systems, the variations are constant, describing particle exchange among different components.

We now return to the diagonalised Hamiltonian, and try to find the lowest-energy states, in the same way as we have done for the example in Sec. IV B. For the moment, we do not impose the constraints, thus regarding all components $\overline{a}_\alpha$ as independent. Only the first r components $\overline{a}_{\tilde\alpha}$ ($\tilde\alpha = 1, \ldots, r$) have a corresponding momentum operator $\overline{P}_{\tilde\alpha}$ in the Hamiltonian, while the other $\kappa - r$ components do not. This means that the resulting states are degenerate in the coordinates $\overline{a}_{r+1}, \ldots, \overline{a}_\kappa$. Thus, we may write the lowest-energy states as

$$\chi_{\mathrm{osc}}(\overline{a}_1, \ldots, \overline{a}_\kappa) = \chi_{\mathrm{osc},r}(\overline{a}_1, \ldots, \overline{a}_r)\,\tilde\chi(\overline{a}_{r+1}, \ldots, \overline{a}_\kappa), \tag{B1}$$

where $\tilde\chi$ is the degenerate part of the wave function (further discussed in Sec. IV B), and

$$\chi_{\mathrm{osc},r}(\overline{a}_1, \ldots, \overline{a}_r) = \exp\!\left(-\frac{e}{2\hbar b}\sum_{\tilde\alpha=1}^{r}\overline{a}_{\tilde\alpha}^\dagger\lambda_{\tilde\alpha}^{-1}\overline{a}_{\tilde\alpha}\right) \tag{B2}$$

is the nondegenerate part. Notice that the components associated with the zero eigenvalues of the matrix E do not contribute. By the observation that the pseudoinverse[28] of $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_r, 0, \ldots, 0)$ is equal to $\widehat{D} = \mathrm{diag}(\lambda_1^{-1}, \ldots, \lambda_r^{-1}, 0, \ldots, 0)$, we may also rewrite $\chi_{\mathrm{osc},r}$ as

$$\chi_{\mathrm{osc},r} = \exp\!\left(-\frac{e}{2\hbar b}\overline{a}^\dagger\widehat{D}\,\overline{a}\right) = \exp\!\left(-\frac{e}{2\hbar b}a^{\circ\dagger}\widehat{K}a^\circ\right),$$

where we used that $\overline{a}^\dagger\widehat{D}\,\overline{a} = a^{\circ\dagger}\widehat{K}a^\circ$. This result is nothing else than Eq. (3.13) with the inverses of D and K replaced by the pseudoinverses. Substituting the density fluctuations $\delta\rho_\alpha$ for the gauge fields a_α° using the constraint (3.3) yields exactly (3.16) by virtue of the property of $\widehat{K}$ that $K\,\widehat{K}\,K = K$. This result is exactly equal to the steps we followed before, but only with K^{-1} replaced by $\widehat{K}$. Therefore, the ground state (3.16) found for the case of strictly positive eigenvalues is also valid if there are zero eigenvalues. All subsequent steps concerning the connection to the trial wave functions remain valid as well.

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[28] The pseudoinverse of a matrix A is the unique matrix $\widehat{A}$ defined by the conditions that $A\widehat{A}$ and $\widehat{A}A$ are hermitian, that $A\widehat{A}A = A$, and that $\widehat{A}A\widehat{A} = \widehat{A}$. This notion is often called Moore–Penrose inverse in literature, see, e.g.: R. Penrose, Proc. Cambridge Philos. Soc. **51**, 406 (1955); A. Ben-Israel and T. N. E. Greville, *Generalized Inverses: Theory and applications*, Pure and Applied Mathematics (Wiley, New York, 1974), ISBN 0-471-06577-3.
[29] Throughout the text, all sums over the component indices $(\alpha, \beta, \ldots)$ are over the values $1, \ldots, \kappa$, unless indicated otherwise.
[30] The given conditions are satisfied simultaneously if M^* is a block-diagonal matrix where the entries within each block are equal to each other, which includes most physically interesting cases.
[31] Other continuous subgroups of $\mathrm{SU}(\kappa)$ may appear as well as symmetry groups. These include product groups like $\mathrm{SU}(2) \times \mathrm{SU}(2)$, which is relevant for four-component systems with two $\mathrm{SU}(2)$ symmetries of a different origin, such as spin and pseudospin.
[32] The $\kappa - r + 1$ complex components u_α, $\alpha = r, \ldots, \kappa$ may indeed be viewed as an element of the complex projective space $\mathrm{CP}^{\kappa-r}$ in which one identifies all elements that differ only by a global (complex) factor c, $(u_r, \ldots, u_\kappa) \equiv (cu_r, \ldots, cu_\kappa)$.
[33] This counterintuitive definition is used in order to have analytic lowest Landau level wave functions. This is due to the negative charge $-e$ of the electrons, for which the basic Hamiltonians are defined. Therefore, one has $\theta(\mathbf{r}) = \arg(x + iy) = \arg z^* = -\arg(z)$.